Entropy and invariant measures of weakly hyperbolic diffeomorphisms

Wandering Seminar, Wroclaw

Lecture 1

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Entropy and invariant measures of diffeomorphisms

Entropy fundamental invariant (Kolmogorov 1958, Shannon 1948)

- counts the orbits
- classifies (Ornstein 1971, Adler-Weiss 1979)

Our goal:

Theorem (Boyle & B 2014)

Any C^{1+}-diffeo of a compact surface is conjugate to a Markov shift, "up to zero entropy invariant probability measures"

Corollary

In particular, such diffeomorphisms with a unique mixing measure maximizing the entropy are classified by their topological entropy

Entropy and invariant measures of diffeomorphisms

Our route for the mini-course:

- positive entropy $\rightsquigarrow$ hyperbolicity in the sense of Pesin
- converse: Katok's horseshoe theorem
- symbolic model for non-uniformly hyperbolic dynamics: Markov shifts (Gurevič)
- hyperbolicity on surfaces $\rightsquigarrow$ Markov shift extension of Sarig
- Markov shifts have a universal property by Hochman

If time allows, study of the invariants:

- (non)existence of measures maximizing the entropy
- value of the entropy (continuity, local constancy, formula)

Plan for lecture 1

- Categories of dynamical systems
- Entropy memento
- Exercises

Categories of dynamical systems

Definition

dynamical system *(***d.s.***)* $=$ *isomorphism of a space up to relevant identifications.*
conjugacy *of d.s.* (X, S), (Y, T) $=$ *isomorphism of spaces* $\psi : X \to Y$ *that* **intertwines** *the maps:* $\psi \circ S = T \circ \psi$
factor $=$ *epimorphism of spaces* $\psi : X \to Y$ *that intertwines the maps.*

- **smooth**: a compact C^r manifold;
- **topological** : a compact metric space (**t.d.s.**);
- **probability**: a probability space up to subsets of zero measure (**p.d.s.**);
- **(standard) Borel**: σ-algebra generated by topology defined by complete distance;
- **almost Borel**: Borel up to subsets with zero measure wrt any aperiodic ergodic invariant probability measure

Categories of dynamical systems

Our focus – **interplay** between these points of view:

- one can forget
- one can enrich, selecting from:
  - $\mathbb{P}(S,X) := \{$ S-invariant Borel probability measures of $X\}$
  - $\mathbb{P}_{\text{erg}}(S,X) := \{\mu \in \mathbb{P}(S,X) : \mu$ S-ergodic$\}$
  - $\mathbb{P}'_{\text{erg}}(S,X) := \{\mu \in \mathbb{P}_{\text{erg}}(S,X) : \mu(\{\text{periodic points}\}) = 0\}$

Theorem (for t.d.s.)

- *(Krylov-Bogolioubov)* $\mathbb{P}(T)$ *and* $\mathbb{P}_{\text{erg}}(T)$ *are not empty*
- *(Dinaburg, Goodman)*

$$h_{\text{top}}(S) = \sup_{\mu \in \mathbb{P}(S)} h(S,\mu) = \sup_{\mu \in \mathbb{P}_{\text{erg}}(S)} h(S,\mu)$$

(definitions are on the way...)

More importantly:**smoothness constrains entropy of measures**

Entropy Memento

Definition

The **(Kolmogorov-Sinai) entropy** *of a p.d.s. is*

$$h(T,\mu) := \sup_P h(T,\mu,P)$$
$$h(T,\mu,P) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}P\right)$$
$$H_\mu(P) := \sum_{A\in P} -\mu(A)\log\mu(A) \quad (0\log 0 = 0)$$

The **(Borel) entropy** *is* $h(T) := \sup_{\mu\in\mathbb{P}_{\mathrm{erg}}(T)} h(T,\mu)$

Proposition

For $\mu \in \mathbb{P}(S)$*, Borel d.s.,* $\exists$ *ergodic decomposition* $\mu = \int \mu_x \, d\mu(x)$*. Then:* $x \mapsto h(T,\mu_x)$ *defined on a Borel set of total* μ*-measure and:*

$$h(T,\mu) = \int h(T,\mu_x)\, d\mu(x).$$

Entropy Memento

Theorem (Goodman-Dinaburg Variational principle)

For a t.d.s. (X, S), $h(S) := \sup_{\mu \in \mathbb{P}_{\text{erg}}(S)} h(S, \mu)$ *is equal to the topological entropy defined as:*

$$h_{\text{top}}(S) := \lim_{\epsilon \to 0} h(S, \epsilon)$$
$$h_{\text{top}}(S, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_S(X, \epsilon, n)$$

where

$$r_S(Y, \epsilon, n) := \min\{\#C \subset Y : Y \subset \bigcup_{x \in C} B_S(x, \epsilon, n)\}$$
$$B_S(x, \epsilon, n) := \{y \in X : \forall 0 \leq k < n \; d(S^k y, S^k x) < \epsilon\}.$$

One can replace $\liminf_{n \to \infty}$ *by* $\limsup_{n \to \infty}$.

Theorem (Katok entropy formula)

If (X, S) *t.d.s.,* $\mu \in \mathbb{P}_{\text{erg}}(S)$, $0 < t < 1$, *then:*

$$h(S, \mu) = \lim_{\epsilon \to 0} \underline{h}_t(S, \mu, \epsilon)$$
$$\underline{h}_t(S, \mu, \epsilon) := \liminf_{n \to \infty} \frac{1}{n} \log \inf_{\mu(Y) > t} r_S(Y, \epsilon, n)$$

One can replace $\liminf_{n \to \infty}$ *by* $\limsup_{n \to \infty}$.

Entropy Memento

Proof (Misiurewicz) of the variational principle: $h_{\text{top}}(S) = h(S)$

Katok formula $\rightsquigarrow h(S) \leq h_{\text{top}}(S)$
Converse: measures equidistributed on minimal (ϵ, n)-covers $\square$

Remark. Limits of μ_ϵ may have small entropy!

Definition

μ **measure maximizing entropy (m.m.e.)** if $h(T, \mu) = h(T)$

Both existence and uniqueness can fail:

Theorem (Misiurewicz 1973)

$\forall 1 \leq r < \infty \ \exists \ f, g \in \text{Diff}^r(\mathbb{T}^4)$:

- *f has no m.m.e.*
- *g has infinitely many m.m.e. (trivial)*

Question **What about dimension 2?**

Theorem (Newhouse 1988)

Any C^∞-diffeomorphism of a compact manifold has some m.m.e.

Exercises

Facts

1. *For* (S, μ) *p.d.s.,* $n \in \mathbb{Z}$*:*
$$h(S^n, \mu) = |n| \cdot h(S, \mu)$$
2. *If* $\pi : (S, \mu) \to (T, \nu)$ *is a factor of p.d.s.* $h(S, \mu) \geq h(T, \nu)$

Exercice

Prove them. Extend them to t.d.s. and Borel d.s.

(Ledrappier-Walters variational principle gives a *formula* for t.d.s.)

Box dimension of compact metric space:
$$\dim_B(X) := \lim_{\epsilon \to 0} \log N(X, \epsilon) / |\log \epsilon|$$
where $N(X, \epsilon) = \min\{\#C : C \subset X \text{ and } \bigcup_{x \in C} B(x, \epsilon) \supset X\}$

Exercice (Kushnirenko 1965)

X *is a compact metric with* $\dim_B X < \infty$
$S : X \to X$ *Lipschitz*
Then $h_{\text{top}}(S) \leq \dim_B X \cdot \log^+(\text{Lip}(S)) \qquad (x^+ := \max(x, 0))$

Exercises

Exercice

Show $h_{\text{top}} = 0$ *for (1) isometries of compact metric spaces; (2) homeomorphisms of the circle.*

(Σ_N, σ_N) *shift on* $\{1, \ldots, N\}$. *Prove/recall:* $h_{\text{top}}(\Sigma_N) := \log N$

A *Misiurewicz* (T, N)*-horseshoe* for $f \in C^0(I)$:

$$\begin{aligned} &\text{open } \emptyset \neq U_1, \ldots, U_N \subset I \text{ s.t.:} \\ &\forall 1 \leq i \neq j \leq N \ \overline{U_i} \cap \overline{U_j} = \emptyset \\ &\forall 1 \leq i, j \leq N \ int(f^T(U_i)) \supset \overline{U_j} \end{aligned}$$

Exercice

Show $K := \bigcap_{n \geq 0} \overline{g^{-n}(U_1 \cup \ldots U_N)}$ *is compact* $g := f^T$*-invariant*

Use $\pi : K \to \Sigma_N$ *s.t.* $g^n x \in \overline{U_{\pi(x)_n}}$ *to show* $h_{\text{top}}(f) \geq \frac{1}{T} \log N$

Generic (residual, fat) = dense G_δ set

(Baire: stable by countable intersection in complete metric spaces)

Exercice (Continued)

Find a generic $\mathcal{G} \subset C^0(I)$ *such that, for all* $g \in \mathcal{G}$, $h_{\text{top}}(g) = \infty$.