The almost Borel structure of surface diffeomorphisms and their spectral decompositions

Jérôme Buzzi (CNRS & Université Paris-Sud)

joint with **M. Boyle**, **S. CROVISIER**, and **O. SARIG**

October 3rd, 2016

Surfaces - CIRM

Outline

Almost Borel conjugacies

$f : X \to X$, $g : Y \to Y$ Borel isomorphisms between Borel spaces

$\mathbb{P}_{\text{erg}}(f)$ set of (ergodic Borel probability) measures
$\mathbb{P}'_{\text{erg}}(f) := \{\mu \in \mathbb{P}_{\text{erg}}(f) : \mu(\{x : \exists n > 0\ f^n(x) = x\}) = 0\}$
$\mathbb{P}^+_{\text{erg}}(f) := \{\mu \in \mathbb{P}_{\text{erg}}(f) : h(f,\mu) > 0\}$ ($h(f,\mu)$ Kolmogorov-Sinai entropy)

Definition

$\psi : X' \to Y'$ **almost Borel conjugacy** : ψ Borel isomorphism ;
$\psi \circ f = g \circ \psi$; $X \setminus X'$, $Y \setminus Y'$ Borel, invariant, "negligible" :
- empty (Borel conjugacy)
- union of periodic orbits (**free Borel conjugacy**)
- negligible for aperiodic measures (**aperiodic Borel conjugacy**)
- negligible for measures with entropy>0 (0-**entropy Borel conjugacy**)

Borel entropy : $h(f) := \sup\{h(f,\mu) : \mu \in \mathbb{P}_{\text{erg}}(f)\}$ invariant
$h(f) = h_{\text{top}}(f)$ if C^0+compact

Theorem (Boyle, B, Gomez)

The topological entropy is a complete invariant of free Borel conjugacy between mixing subshifts of finite type

Symbolic dynamics

Markov shift $\Sigma = \Sigma(\mathcal{G})$ for $\mathcal{G} \subset \mathcal{A} \times \mathcal{A}$, countable oriented graph :

- $\Sigma := \{x \in \mathcal{A}^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ x_n \xrightarrow{G} x_{n+1}\}$
- $\sigma : \Sigma \to \Sigma,\ (x_n)_{n\in\mathbb{Z}} \to (x_{n+1})_{n\in\mathbb{Z}}$

For this talk : $h(\Sigma) < \infty$

Remark If $\mathcal{G}$ finite, $\Sigma(\mathcal{G})$ subshift of finite type

Irreducible components of Σ :

- for $a, b \in \mathcal{A}$: $a \leftrightarrow b \iff \exists$ paths from a to b and from b to a
- irreducible components of Σ : $\Sigma_{\mathcal{C}} := \Sigma(\mathcal{G}|\mathcal{C} \times \mathcal{C})$ for $\mathcal{C} \in \mathcal{A}/_{\leftrightarrow}$
- $h(\mathcal{C}) := h(\Sigma_{\mathcal{C}})$
- $\text{per}(\mathcal{C}) := \gcd\{n \geq 1 : \exists x \in \Sigma_{\mathcal{C}}\ \sigma^n x = x\}$

Theorem (Gurevic 1971)
For each $\mathcal{C}$, at most one **local mme** *: $h(\sigma, \mu) = h(\mathcal{C})$*
Moreover, (f, μ) conjugate to (Bernoulli with entropy $h(\mathcal{C})$) $\times\ \mathbb{Z}/\mathbb{Z}_{\text{per}(\mathcal{C})}$

Spectral decomposition

$\Sigma(\mathcal{G}) = \sqcup_{\mathcal{C} \in \mathcal{A}/_{\leftrightarrow}} \Sigma_{\mathcal{C}}$ partition up to points nonrecurrent in future or past

In particular : $\forall \mu \in \mathbb{P}_{\text{erg}}(\Sigma)\ \exists! \mathcal{C} \in \mathcal{A}/_{\leftrightarrow}$ such that $\mu(\Sigma_{\mathcal{C}}) = 1$

Symbolic dynamics for surface diffeomorphisms

Let $f \in \mathrm{Diff}^{1+}(S)$, S closed surface

$\Sigma^{\#} := \{x \in \Sigma : \exists a, b \in \mathcal{A} : \{n \leq 0 : x_n = a\}, \{n \geq 0 : x_n = b\} \text{ infinite}\}$

Theorem (Sarig 2014)

For $\chi > 0$, $\exists$ Markov shift Σ_χ and $\pi_\chi : \Sigma_\chi \to S$ such that :
- *π_χ Hölder and $\pi_\chi \circ \sigma = f \circ \pi_\chi$;*
- *$\forall \mu \in \mathbb{P}_{\mathrm{erg}}(f)$ $h(f, \mu) > \chi \implies \mu(\pi_\chi(\Sigma_\chi^{\#})) = 1$;*
- *$\forall x \in S$ $\pi^{-1}(x) \cap \Sigma_\chi^{\#}$ finite*

Corollaries

- *If $h_{\mathrm{top}}(f) > 0$, there are countably many mme's*
- *If $h_{\mathrm{top}}(f) > 0$ and if there is some mme,*
  $\exists p \in \mathbb{N}^* \ \liminf_{n\to\infty, p|n} e^{-nh_{\mathrm{top}}(f)} \cdot \#\{x \in S : f^n(x) = x\} > 0$

Lemma-Definition (Boyle-B, to appear)

π_χ above is a **Bowen factor** *:*

$$\forall x, y \in \Sigma_\chi^{\#} \ \pi(x) = \pi(y) \iff \forall n \in \mathbb{Z} \ x_n \sim y_n$$

Theorem (Boyle-B, to appear)

f is 0-entropy Borel conjugate to a Markov shift

An invariant for Markov shifts

Underlying : theorem of **Hochman** (2014 +) for mixing Markov shifts

Σ Markov shift with $h(\Sigma) < \infty$

Recall spectral decomposition :
$\Sigma = \bigsqcup_{\mathcal{C} \in \mathcal{A}/\leftrightarrow} \Sigma_{\mathcal{C}}$

Entropy at period $p \in \mathbb{N}^*$:
$h(p) := \sup\{h(\mathcal{C}) : \mathrm{per}(\mathcal{C}) | p\}$

Visible periods :
$\mathcal{P}(\Sigma) := \{p \in \mathbb{N}^* : h(p) > \sup^+\{h(q) : q|p,\ q \neq p\}\}$
$\cup\{\mathrm{per}(\mathcal{C}) : h(\mathcal{C}) = \sup^+\{h(q) : q|\mathrm{per}(\mathcal{C})\}$ and $\Sigma_{\mathcal{C}}$ with local mme$\}$

Entropy-period spectrum : (H, M)

- $H : P(\Sigma) \to [0, \infty)$, $H(p) := h(p)$
- $M : P(\Sigma) \to \bar{\mathbb{N}}$, $M(p) := \#\{\mathcal{C} : \mathrm{Per}(\mathcal{C}) = p,\ h(\mathcal{C}) = h(p)$, with local mme$\}$

The almost-Borel classification of Markov shifts

Recall :

- $h(p) := \sup\{h(\mathcal{C}) : \mathrm{per}(\mathcal{C})|p\}$
- $\mathcal{P}(\Sigma) := \{p \in \mathbb{N}^* : h(p) > \sup^+\{h(q) : q|p,\ q \neq p\}\}$
  $\cup\{\mathrm{per}(\mathcal{C}) : h(\mathcal{C}) = \sup^+\{h(q) : q|\mathrm{per}(\mathcal{C})\}$ and $\Sigma_{\mathcal{C}}$ with local mme$\}$
- $H : P(\Sigma) \to [0, \infty)$, $H(p) := h(p)$
- $M : P(\Sigma) \to \bar{\mathbb{N}}$, $M(p) := \#\{\mathcal{C} : \mathrm{Per}(\mathcal{C}) = p,\ h(\mathcal{C}) = h(p)$, with local mme$\}$

Theorem (Boyle, B (to appear))

(H, M) *complete invariant for free Borel conjugacy among Markov shifts*

Remarks

1. $(h(\mathcal{C}), \mathrm{per}(\mathcal{C})$, existence of local mme on $\Sigma_{\mathcal{C}})_{\mathcal{C} \in \mathcal{A}/\leftrightarrow}$ too detailed
2. $\Sigma \approx \bigsqcup_{p \in P(\Sigma)} \Sigma_{H(p), M(p)}$ "smallest" Markov shift in the class
3. Characterization :
   $H(p) = \sup_{q|p} H(q)$ and $\left[M(p) = 0 \implies \forall q|p, q \neq p : H(p) > H(q)\right]$
   with $\sup H < \infty$

Almost-Borel classification of surface diffeomorphisms

Problem :
Classify smooth surface diffeos up 0-entropy Borel conjugacy

Smooth = C^{1+} or C^{∞}

Strategy :

1. Reduced to Markov shifts (Boyle-B., above)
2. Complete invariant for Markov shifts with finite entropy (Boyle-B, above)
3. Understand the conjugacy classes in terms of surface dynamics (Crovisier-B-Sarig, to be described)
4. *Understand which Markov shifts are realized by surface diffeos*

Homoclinic classes of periodic orbits

For $\mathcal{O} \in HPO(f)$ the set of **hyperbolic periodic orbits**

$W^\sigma(\mathcal{O}) := \bigcup_{x\in\mathcal{O}} W^\sigma(x) \qquad \sigma = s, u$

$\mathcal{O} \sim \mathcal{O}'$ iff $W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O}') \neq \emptyset$, $W^u(\mathcal{O}) \pitchfork W^s(\mathcal{O}') \neq \emptyset$

The **homoclinic class** is :

$H(\mathcal{O}) := \overline{\{\mathcal{O}' : \mathcal{O}' \sim \mathcal{O}\}}$

Fact $\sim$ *is an equivalence relation over* $HPO(f)$

λ-lemma

T submanifold transverse to $W^s(\mathcal{O})$

K is a compact submanifold contained in $W^u(\mathcal{O})$

Then $\exists T_n \subset T$, $k_n \to \infty$, s.t. $f^{k_n}(T_n) \to K$ (C^1-topology)

Lemma

$HC(\mathcal{O})$ is transitive, invariant compact set

For $p \in \mathcal{O}$,

$H(\mathcal{O}) = \bigcup_{j=0}^{T-1} f^j K$ *with* $K := \overline{W^s(p) \pitchfork W^u(p)}$ *and* $f^T(K) = K$

Spectral decomposition for surface diffeomorphisms

Theorem (B-Crovisier-Sarig)

For $f \in \mathrm{Diff}^\infty(M^2)$, there are hyperbolic periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots$ s.t.

- *$\bigcup_{n\geq 1} HC(\mathcal{O}_n)$ has full measure for all $\mu \in \mathbb{P}^+_{\mathrm{erg}}(f)$*
- *each $HC(\mathcal{O}_n)$ 0-entropy Borel conjugate to irreducible Markov shift*
- *each $HC(\mathcal{O}_n)$ has a unique local mme μ_n Bernoulli$\times\mathbb{Z}/_{p_n}\mathbb{Z}$*

Moreover,

- $n \neq m \implies h_{\mathrm{top}}(f, HC(\mathcal{O}_n) \cap HC(\mathcal{O}_m)) = 0$
- $\lim_{n\to\infty} h_{\mathrm{top}}(HC(\mathcal{O}_n)) = 0$

Theorem (in preparation)

$$\liminf_{n\to\infty, p_n | n} e^{-n\cdot h_{\mathrm{top}}(HC(\mathcal{O}_n))} \cdot \#\{x \in HC(\mathcal{O}_n) : f^n x = x,\ \textit{hyperbolic}\} \geq p_n$$

The entropy-period spectrum for surface diffeomorphisms

For each $HC(\mathcal{O}_n)$, let :

- $h_n := h_{\text{top}}(HC(\mathcal{O}_n))$;
- $p_n \geq 1$ such that μ_n Bernoulli$\times\mathbb{Z}/_{p_n}\mathbb{Z}$

Corollary The entropy-period spectrum (H, M) can be expressed as

- For $p \in \mathbb{N}^*$, $h(p) := \sup\{h_{\text{top}}(HC(\mathcal{O}_n)) : p_n = p\}$
- $P(f) := \{p \in \mathbb{N}^* : h(p) = \sup_{q|p} h(q) > 0\}$
- $H : P(f) \to [0, h_{\text{top}}(f)]$, $H(p) := h(p)$
- $M : P(f) \to \mathbb{N}^*$, $M(p) := \#\{n \geq 1 : h_{\text{top}}(HC(\mathcal{O}_n)) = h(p),\ p_n = p\}$

Qualitative obstructions wrt Markov shifts with finite entropy :

- $0 < M(p) < \infty$ for each $p \in P(f)$
- $\lim_{p\to\infty} H(p) = 0$

Remark (H, M) from *measures maximizing entropy at a period* (mmePer) :
$\mu \in \mathbb{P}^+_{\text{erg}}(f)$ s.t. $\forall \nu \in \mathbb{P}^+_{\text{erg}}(f)$ $\text{Per}(f,\nu) \subset \text{Per}(f,\mu) \implies h(\nu) \leq h(\mu)$
- $P(f) := \{\max \text{Per}(f,\mu) : \mu \text{ mmePer}\}$
- $H : P(f) \to [0, h_{\text{top}}(f)]$, $H(p) := h(\mu)$ for any mmePer with $\max \text{Per}(\mu) = p$
- $M : P(f) \to \mathbb{N}^*$, $M(p) := \#$ mmePer with with $\max \text{Per}(\mu) = p$

Conclusion

S is a closed surface

We have :

- a complete invariant (H, M) for 0-entropy Borel conjugacy over $\mathrm{Diff}^\infty(S)$, S closed surface
- a "somewhat" concrete expression for (H, M)
- qualitative obstructions on a Markov shift to be realized in $\mathrm{Diff}^\infty(S)$

but we really want :

- quantitative obstruction
- characterization

A way forward ?

For natural extensions of *interval maps* : quantitative obstructions from pattern forcing theory (Sharkovskii, Misiurewicz,...)
(Work in progress with **S. Ruette**, characterization for $\#\{x : f'(x) = 0\} < \infty$)

For surface diffeomorphisms, forcing pattern theory of Franks, Misiurewicz, Los,..., depend on isotopic information

Help !

but for now **thank you**