Entropy of C^1-diffeomorphisms without dominated splitting

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Outline

Understanding topological entropy

$f : M \to M$ compact, connected, without boundary

Topological Entropy (Adler-McAndrew-Konheim 1968)

$$h_{\text{top}} : \text{Diff}^r(M) \to [0, \infty)$$

How does it vary?

- continuity: Misiurewicz,Katok (low d); Newhouse,Yomdin C^∞
- local constancy: stability beyond hyperbolicity (B-Fisher)
- robust instability (diffeos not approximated by local constancy)?

What are its sources?

- homology: Shub's Entropy conjecture
- volume growth: Yomdin, Newhouse C^∞
- combinatorics through Markov partitions: Bowen
- combinatorics through horseshoes: Katok $\text{Diff}^{1+\alpha}(M^2)$

How does it classify?

- Generators: Jewett-Krieger, Hochman, Burguet-Downarowicz
- Almost ... conjugacy: Adler-Marcus, Boyle-B-Gomez

Which values does it takes?

Topological and measure entropies - Definitions

$f : M \to M$ C^0, compact, $\mu \in \mathbb{P}_{\text{erg}}(f)$

Topological Entropy (Adler-McAndrew-Konheim 1965; Bowen 1971)

$h_{\text{top}}(f) = \lim_{\epsilon \to 0+} h_{\text{top}}(f, \epsilon)$

$h_{\text{top}}(f, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_f(\epsilon, n, M)$

Measured Entropy (Kolmogorov-Sinai 1958; Katok 1980)

$h(f, \mu) = \lim_{\epsilon \to 0+} h(f, \mu, \epsilon)$

$h(f, \mu, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_f(\epsilon, n, \mu)$

Tail Entropy (Misiurewicz-Bowen 1973)

$h^*(f) = \lim_{\epsilon \to 0+} h^*(f, \epsilon)$

$h^*(f, \epsilon) = \sup_{x \in M} h_{\text{top}}(f, B_f(x, \epsilon, \infty)) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in M} r_f(\delta, n, B_f(x, \epsilon, n))$

Variational principle (Goodman 1971)

$h_{\text{top}}(f) = \sup\{h(f, \mu) : \mu \in \mathbb{P}_{\text{erg}}(f)\}$

Measure maximizing the entropy (mme)

$\mu_{\max} \in \mathbb{P}_{\text{erg}}(f)$ with $h(f, \mu_{\max}) = \sup\{h(f, \mu) : \mu \in \mathbb{P}_{\text{erg}}(f)\}$

Newhouse: $C^\infty \implies$ existence

Main Theorem

For $\mu \in \mathbb{P}_{\text{erg}}(f)$, $f \in \text{Diff}^1(M)$, M closed

Lyapunov Exponents $\lambda_1(f,\mu) \leq \lambda_2(f,\mu) \leq \cdots \leq \lambda_d(f,\mu)$

Ruelle's inequality: $h(f,\mu) \leq \Delta(f,\mu) := \min\left(\sum_i \lambda_i(f,\mu)^+, \sum_i \lambda_i(f,\mu)^-\right)$

Main Theorem (B-Crovisier-Fisher)

$\mathcal{U}$ a neighborhood of f in $\text{Diff}^1(M)$, $\text{Diff}^1_{\text{vol}}(M)$ or $\text{Diff}^1_\omega(M)$
$\mathcal{O}$ periodic orbit with large period and **no strong dominated splitting**
Then, for each $U \supset \mathcal{O}$, there is a horseshoe $\mathcal{O} \subset K \subset U$ for $g \in \mathcal{U}$ s.t.

$$h_{\text{top}}(g,K) \geq \Delta(g,\mathcal{O}) = \Delta(f,\mathcal{O})$$

Moreover: $\{g \neq f\} \subset U \setminus \mathcal{O}$; can preserve a homoclinic relation

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Remark "Optimal": $\limsup_{g \to f} h_{\text{top}}(g) \leq \sup_\mu \Delta(f,\mu)$

Remark Specific to C^1-topology: tools; factor $1/r$ in C^r
Locally uniform bounds on required period, domination

- Newhouse 1978 ($d = 2$);
- Catalan-Tahzibi 2014 (symplectic, entropy $\min_i(|\lambda_i(f,\mathcal{O})|)$)

(see also: Catalan 2016)

Tools for localized perturbations

- Perturbations of periodic linear cocycles (Bochi-Bonatti, Gourmelon, new for symplectic)
  - making spectrum simple with rational angles;
  - mixing the stable (unstable) exponents
  - creating a small angle

- Local support with homoclinic connection, form or symplectic form (Gourmelon):
  - Franks' Lemma with linearization (Avila for volume-preserving)
  - Homoclinic tangency from lack of domination (Gourmelon)

"Localized perturbations of conservative C^1 diffeomorphisms",
`arxiv:1612:06914`

Proof - Part I: circular permutation

$f \in \mathrm{Diff}^1(M)$ with $\mathcal{O}(p)$ a long periodic orbit with weak domination

Use previous tools to create, by perturbations:

1. Transverse homoclinic point and locally linear horseshoe $K_0 \ni p$

2. Point $x \in K_0$ with large period and $Df^{\pi(x)}|_x = \Lambda^s \mathrm{Id}_{E^s} \times \Lambda^u \mathrm{Id}_{E^u}$

3. Homoclinic tangency z for $\mathcal{O}(x)$

4. Linearize around $\mathcal{O}(x)$ so $\exists$ loc. invariant $\underbrace{E_1 \oplus \dots E_k}_{E^s} \oplus \underbrace{E_{k+1} \oplus \dots E_d}_{E^u}$

5. With $F = T_z W^s(x) \cap T_z W^u(x)$ assumed to be 1-d:

$$T_z W^s(x) = F \oplus F_1^s \oplus \dots \oplus F_{k-1}^s$$
$$T_z W^u(x) = F_1^u \oplus \dots \oplus F_{k-d-1}^u \oplus F$$
$$T_z M = F \oplus (F_1^s \oplus \dots \oplus F_{k-1}^s) \oplus G \oplus (F_1^u \oplus \dots \oplus F_{k-d-1}^u)$$

6. Using $Df^{\pi(x)}$ =homothety$\times$homothety and $F' \oplus F_1^u \oplus F_{k-d-1}^u \pitchfork E^s$:

Perturb future of z to get $F \to E_1$, $F_i^s \to E_{i+1}$
and $G \to E_{k+1}$, $F_j^u \to E_{k+1+j}$

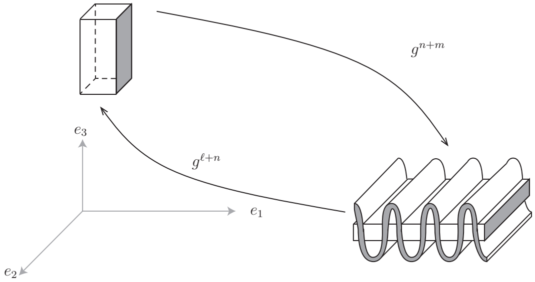
Similarly in the past $F_i^s \to E_i$, $G \to E_k$, $F_j^u \to E_{k+j}$, $F \to E_d$

Conclusion

$\exists \tau \in W^u_{\mathrm{loc}}(x) \cap f^{-m} W^s_{\mathrm{loc}}(x)$ s.t. $Df^m|_\tau . E_i = E_{i+1} \quad (E_{d+1} = E_1)$

Proof - Part II: entropy from exponents

Case $d = 3$, $k = 2$ $(\lambda_1 < \lambda_2 < 0 < \lambda_3)$



$\delta_1 = \delta$, $\delta_2 = e^{n\lambda_1}$, $\delta_3 = \delta e^{-\lambda_3} n$
Image by f^n has height along e_1: $\delta e^{(\lambda_1+\lambda_2)n}$

Wiggles $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_k, x_{k+1} + H\cos(\pi N x_1), x_{k+2}, \ldots, x_d)$

- to cross: $H \geq C(e^{(\lambda_1+\lambda_2)n} + e^{-\lambda_3 n})\delta$
- to be C^1-small: $N = o(H^{-1})$

Entropy $\frac{\log N}{n} \preceq \max(-\lambda_1 - \lambda_2, \lambda_3) = \Delta(f, \mathcal{O}(x))$ $\square$

Application 1: C^1 horseshoes from LACK of domination

Let $f \in \mathrm{Diff}^1(M)$ be **generic** (ie, belonging to dense G_δ in $\mathrm{Diff}^1(M)$)

Theorem (B-Crovisier-Fisher)

For any $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$, if $\mathrm{supp}\mu$ has no dominated splitting, then are horseshoes K_n approximating μ:
(i) in entropy;
(ii) in Hausdorff distance;
(iii) in weak-star topology

Compare Katok C^{1+}; Gan, Gelfert C^1 + adapted dominated splitting

Remark Does not say that $K_n \subset \mathrm{supp}(\mu)$ or homoclinically related

Ingredients of the proof:

- Ergodic closing lemma with control of exponents
- Main theorem

Application 2: Infinitely many homoclinic classes

Homoclinic relation and classes for hyperbolic periodic orbit $\mathcal{O}$:

$\mathcal{O} \sim \mathcal{O}' \iff W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O}')$ et $W^u(\mathcal{O}) \pitchfork W^s(\mathcal{O}')$

$HC(\mathcal{O}) := \overline{\bigcup_{\mathcal{O}' \sim \mathcal{O}} \mathcal{O}'}$ compact, invariant, transitive

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Theorem (B-Crovisier-Fisher)

$f \in \mathrm{Diff}^1(M)$ **generic**

Any $HC(\mathcal{O})$ without dominated splitting is accumulated by **infinitely many** *homoclinic classes with entropy bounded away from zero*

More precisely, $\liminf_n h_{\mathrm{top}}(HC(\mathcal{O}_n)) \geq \sup_{\mathcal{O}' \sim \mathcal{O}} \Delta(f, \mathcal{O}')$

Remark Newhouse's theorem would suffice (smaller bound)

Ingredients:

- $\mathcal{O}' \sim \mathcal{O}$ with $\Delta(\mathcal{O}') > \Delta(\mathcal{O})$, long period, weak domination
- Franks Lemma and linear perturbation to make $\mathcal{O}'$ a sink/source
- undo the perturbation inside the basin
- Main Theorem

Application 3: Entropy formulas in conservative settings

M closed manifold with dim $d \geq 2$, ω volume or symplectic form

Let $\mathcal{E}^1_\omega(M) := \mathrm{int}(\{f \in \mathrm{Diff}^1_\omega(M) : \text{no domination}\})$

Theorem (B-Crovisier-Fisher)

The topological entropy of a generic $f \in \mathcal{E}^1_\omega(M)$ is equal to:

(1) $\sup\{h_{\mathrm{top}}(f, K) : K \textit{ horseshoe}\}$

(2) $\sup\{\Delta(f, \mathcal{O}) : \mathcal{O} \textit{ periodic orbit}\}$

(3) $\max_{0<k<d} \lim_{n\to\infty} \frac{1}{n} \log \sup_{E \in G_k(TM)} Jac(f^n, E)$

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generalizes, strengthens Catalan-Tahzibi (2014)

Ingredients of the proof

- $\sup_K h_{\mathrm{top}}(K) \leq h_{\mathrm{top}}(f) = \sup_\mu h(\mu) \leq \sup_\mu \Delta(\mu)$ (always)
- erg. measures $\approx$ arb. dense periodic orbits (generic, Abdenur-Bonatti-Crovisier)

$\rightsquigarrow \sup_\mu \Delta(\mu) = \Delta(f) := \sup_{\mathcal{O}} \Delta(f, \mathcal{O})$
- Δ is continuous at generic diffeo
- $\sup_K h_{\mathrm{top}}(K) > \Delta(f) - \epsilon$ open and dense

Application 4: Instability of the entropy

M closed manifold with dim $d \geq 2$, ω volume or symplectic form

Let $\mathcal{E}^1_\omega(M) := \text{int}(\{f \in \text{Diff}^1_\omega(M) : \text{no domination}\})$

Theorem (B-Crovisier-Fisher)

The topological entropy of a generic $f \in \mathcal{E}^1_\omega(M)$ is equal to:

(1) $\sup\{h_{\text{top}}(f, K) : K \textit{ horseshoe}\}$

(2) $\sup\{\Delta(f, \mathcal{O}) : \mathcal{O} \textit{ periodic orbit}\}$

(3) $\max_{0<k<d} \lim_{n\to\infty} \frac{1}{n} \log \sup_{E \in G_k(TM)} Jac(f^n, E)$

Corollary h_{top} is nowhere locally constant in $\mathcal{E}^1_\omega(M)$ (**robust instability**)

Corollary For any dense G_δ $\mathcal{G} \subset \mathcal{E}^1_\omega(M)$, $h_{\text{top}}(\mathcal{G})$ **uncountable**

Corollary Generic $f \in \mathcal{E}^1_\omega(M)$ is a **continuity point** of $h_{\text{top}}|\text{Diff}^1_\omega(M)$

Corollary C^1 generically : no domination $\iff$ $h^*(f) = h_{\text{top}}(f)$

Application 5: No mme and Borel classification

M closed manifold with dim $d \geq 2$, ω volume or symplectic form

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Theorem (B-Crovisier-Fisher)

Generic $f \in \mathcal{E}^1_\omega(M)$ has no measure maximizing the entropy

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Remark The diffeos with m.m.e. are **dense** (Newhouse theorem for C^∞)

Combining horseshoes, no m.m.e. and Hochman (arxiv 2015):

Corollary (B-C-F)

There is dense G_δ subset of $\mathcal{E}^1_\omega(M)$ among which the **topological entropy is a complete invariant** *for Borel conjugacy after removing periodic points*

Proof of no m.m.e. - Concentration phenomenon

$f \in \mathcal{E}^1_\omega(M)$, $\dim M \geq 2$

Dynamical ball for $x \in M$, $\epsilon > 0$:

$$B_f(x, \epsilon, n) := \{y \in M : \forall 0 \leq k < n \; d(f^k y, f^k x) < \epsilon\}$$

Proposition

$\forall 0 < \epsilon, \alpha < 1$, for a dense set of $f_0 \in \mathcal{E}^1_\omega(M)$, $\exists \delta > 0$, $\exists$ finite $X \subset M$ s.t.

(*) if f close to f_0, $\mu \in \mathbb{P}_{\text{erg}}(f)$, and $h(f, \mu) > h_{\text{top}}(f) - \delta$,
then $\mu(\bigcup_{x \in X} B_f(x, \epsilon, \#X)) > 1 - \alpha$

Proof of Theorem.

1) $\exists \mathcal{G}$ dense $G_\delta \subset \mathcal{E}^1_\omega(M)$ s.t. $\forall 0 < \epsilon, \alpha < 1 \; \forall f \in \mathcal{G} \; \exists \delta > 0 \; \exists X$ finite satisfying (*)

2) Let $f \in \mathcal{G}$, μ m.m.e., and $\epsilon > 0$.
From Katok's formula, need to bound:

$$r_f(\mu, \epsilon, n) := \min\{\#C : \mu(\bigcup_{x \in C} B_f(x, \epsilon, n)) > 1/2\}$$

3) Take $0 < \alpha << 1/\log \min\{\#C : \bigcup_{x \in C} B(x, \epsilon) = M\}$. Apply (*)

□

Conclusion

Conjecture (higher smoothness)
Given a C^r-diffeo with hyperbolic periodic point p in a cycle of basic sets (see Gourmelon) with no dominated splitting, there is a C^r-perturbation with a horseshoe with entropy $\geq \Delta(p)/r$

Question (internal perturbations)
For a homoclinic class of a C^1-generic diffeo, is the topological entropy the supremum of that of the horseshoes it contains?

Question (entropy instability)
Show that $\{f \in \mathrm{Diff}^1(M) : h_{\mathrm{top}}$ not locally constant at $f\}$ has non-empty interior

Problem (entropy stability)
Characterize the locus of entropy stability

$$\overline{\bigcup\{U \text{ open in } \mathrm{Diff}^1_\omega(M) : h_{\mathrm{top}}|U = \mathrm{const}\}}$$

(Generically in $\mathrm{Diff}^1_\omega(M)$: no domination $\iff h^* = h_{\mathrm{top}}$)

`arxiv:1606.01765`, `arxiv:1612:06914`
JB, S. Crovisier, T. Fisher, The entropy of C^1-diffeomorphisms without a dominated splitting

JB, S. Crovisier, T. Fisher, Local perturbations of conservative C^1 diffeomorphisms