Strong Positive Recurrence for Diffeomorphisms

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Beyond Uniform Hyperbolicity in Bedlewo

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Outline

Spectral gap: Exponential mixing for smooth surface diffeomorphisms

$f \in \mathrm{Diff}^r(M)$ with M a closed d-dimensional manifold

Classical variational principle: $h_{\mathrm{top}}(f) = \sup_{\mu \in \mathbb{P}_{\mathrm{erg}}(f)} h(\mu)$

A measure maximizing the entropy (MME) is $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(\mu) = h_{\mathrm{top}}(f)$

If $r = \infty$, some MME exists (Newhouse)
If $r = \infty$, $d = 2$, $h_{\mathrm{top}}(f) > 0$, and top. transitive, unique MME (B-Crovisier-Sarig)

Theorem (B.-Crovisier-Sarig)

Assume that $r = \infty$, $d = 2$, topological mixing, and $h_{\mathrm{top}}(f) > 0$.

Then the unique MME μ is exponentially mixing for Hölder functions, ie

For any exponent $0 < \alpha \leq 1$, there is $\kappa < 1$ such that

$$\forall u, v \in C^\alpha(M)\ \int u \circ f^n \cdot v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)$$

Goal

A notion of hyperbolicity:
- sufficiently relaxed to hold beyond uniform hyperbolicity
- strong enough to retain "spectral gap"

Nonuniformly hyperbolic measures

$f \in \mathrm{Diff}^r(M^d)$ with $r > 1$ and M a closed d-dim. Riemannian manifold

$\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ ergodic (invariant Borel probability) measure

Definition

Lyapunov spectrum: $\sigma_L(x) := \{\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n.v\| : v \in T_xM \setminus 0\}$

Oseledets spaces: $E_x^\lambda := \{v \in T_xM \setminus 0 : \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda\}$

Lyapunov exponents: $\lambda^1(x) \geq \lambda^2(x) \geq \cdots \geq \lambda^d(x)$

For $\mu \in \mathbb{P}(f)$, $\lambda^j(\mu) := \int \lambda^j(x)\, d\mu$

Definition (Pesin hyperbolicity)

$\nu \in \mathbb{P}(f)$ is **hyperbolic** if ν-ae $0 \notin \sigma_L(x)$

$\nu \in \mathbb{P}(f)$ is hyperbolic of **saddle type** if additionally ν-ae $\lambda^1(x) > 0 > \lambda^d(x)$

Key observation (Katok)

If $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$, $h(\mu) > 0$ and $d = 2$, Ruelle's inequality shows μ hyperbolic saddle type

More precisely, $\lambda^1(\mu) \geq h(\mu) > 0 > -h(\mu) \geq \lambda^2(\mu)$

Pesin theory of nonuniform hyperbolicity

Definition

For $0 < \epsilon < \chi$ and $C \geq 1$, the Pesin block $\Lambda_\chi(C,\epsilon)$ is the set of $x \in M$ for which there is $T_xM = E \oplus F$ satisfying:

$$\forall n \geq 0\ \forall k \in \mathbb{Z} \quad \|Df^n|_{Df^k(E)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$
$$\forall n \geq 0\ \forall k \in \mathbb{Z} \quad \|Df^{-n}|_{Df^k(F)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$

Fact

$\Lambda_\chi(C,\epsilon)$ is compact with $f^{\pm 1}(\Lambda_\chi(C,\epsilon)) \subset \Lambda_\chi(Ce^\epsilon,\epsilon)$

Invariant $K \Subset M$ is hyperbolic iff $K \subset \Lambda_\chi(C,\epsilon)$ for some $\chi > 0$ and arb. small $\epsilon > 0$.

Lemma (Oseledets-Pesin reduction)

If $\mu \in \mathbb{P}(f)$ χ-hyperbolic ie μ-ae $\{\lambda^j(x) : j = 1, \ldots, d\} \cap [-\chi, \chi] = \emptyset$ then

$$\forall \epsilon > 0\ \exists C > 1\ \mu(\Lambda_\chi(C,\epsilon)) > 1 - \epsilon$$

Theorem (Pesin local invariant manifolds)

For $\chi > 0$ and $0 < \epsilon \leq \epsilon(f,\chi)$, $W^s(x), W^u(x)$ are C^r-immersions, C^0 on $x \in \Lambda_\chi(C,\epsilon)$

SPR property for diffeomorphisms

$f \in \mathrm{Diff}^r(M^d)$ with $r > 1$ and M closed d-dimensional manifold ($d \geq 2$)

Definition

For $0 < \epsilon < \chi$ and $C \geq 1$, the Pesin block $\Lambda_\chi(C, \epsilon)$ is the set of $x \in M$ for which there is $T_x M = E \oplus F$ satisfying:

$$\forall n \geq 0\ \forall k \in \mathbb{Z} \quad \|Df^n|_{Df^k(E)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$
$$\forall n \geq 0\ \forall k \in \mathbb{Z} \quad \|Df^{-n}|_{Df^k(F)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$

Definition (B-Crovisier-Sarig)

$f \in \mathrm{Diff}^r(M^d)$ is **strongly positively recurrent** (or **SPR**) if there is $\chi > 0$ such that

For any $\epsilon > 0$, there are $h < h_{\mathrm{top}}(f)$, $C > 1$, and $\tau > 0$ such that

$$\forall \mu \in \mathbb{P}_{\mathrm{erg}}(f)\ h(\mu) > h \implies \mu(\Lambda_\chi(C, \epsilon)) > \tau$$

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^\infty(M^2)$ with M^2 closed surface

If $h_{\mathrm{top}}(f) > 0$, then f is strongly positively recurrent

Statistical properties

Theorem (B-Crovisier-Sarig)
Let $f \in \mathrm{Diff}^{1+}(M^d)$ *be SPR with MME* μ *(*$\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ *and* $h(\mu) = h_{\mathrm{top}}(f)$*)*
If μ *is strongly mixing then it is* exponentially mixing *for Hölder functions, ie*

For any exponent $0 < \alpha \leq 1$*, there is* $\kappa < 1$ *such that*

$$\forall u, v \in C^\alpha(M) \int u \circ f^n \cdot v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)$$

Theorem (B-Crovisier-Sarig)
Let $f \in \mathrm{Diff}^{1+}(M^d)$ *be SPR with* $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ *such that* $h(\mu) = h_{\mathrm{top}}(f)$
If μ *is strongly mixing then, for Hölder-continuous functions we have:*

- *Central limit theorem (CLT)*
- *Identification of the variance and characterization of its vanishing*
- *Large deviations*
- *Almost sure invariance principle and its consequences (law of iterated logarithm, arcsine law, law of records)*

Remark. There are statements for general (ie periodic) ergodic MMEs

MMEs of SPR diffeomorphisms: existence and finiteness

Theorem (B-Crovisier-Sarig)
Let $f \in \mathrm{Diff}^{1+}(M^d)$ *be SPR with* $h_{\mathrm{top}}(f) > 0$
If f *is SPR then* f *admits a nonzero, finite number of ergodic MMEs* $m_1, \dots, m_K$

Remark. New proof of the finiteness of MMEs [B-Crovisier-Sarig Annals 2022]

Theorem (Effective intrinsic ergodicity, B-Crovisier-Sarig)
Let $f \in \mathrm{Diff}^{1+}(M^d)$ *be SPR with* $h_{\mathrm{top}}(f) > 0$ *with a unique MME* m
given $0 < \alpha \leq 1$*, there is* $C > 0$ *such that for any* $\mu \in \mathbb{P}_{\mathrm{erg}}(f$*,*

$$\forall u \in C^\alpha(M) \ \left|\int u \, d\mu - \int u \, dm\right| \leq C \|u\|_{C^\alpha} \sqrt{h_{\mathrm{top}}(f) - h(\mu)}$$

Remark.
- Effective intrinsic ergodicity is a quantitative version of the softer

  USC of $h : \mathbb{P}(f) \to \mathbb{R}$ and $\exists$ unique MME m gives $\mu \xrightarrow{*} m$ as $h(\mu) \to h_{\mathrm{top}}(f)$
- Proved for SFT by [S. Kadyrov 2015]; for Markov shifts by [Rühr-Sarig, arxiv]

MMEs of SPR diffeomorphisms: Lyapunov exponents

$\lambda^1(x) \geq \cdots \geq \lambda^d(x)$: pointwise Lyapunov exponents (repeated according to multiplicity)
$\lambda^j(\mu) := \int_X \lambda^j(x)\, d\mu$: average Lyapunov exponents (repeated according to multiplicity)
$J^u(\mu) := \sum_{j=1}^d \max(\lambda^j(\mu), 0)$: sum of positive exponents

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^d)$ of a closed manifold

f is SPR if and only if $\exists \chi > 0$ such hat the following holds:

For any $\mu_k \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(\mu_k) \to h_{\mathrm{top}}(f)$ and $\exists \lim_n \mu_n =: \mu$

1. *$\exists i = i(\mu)$ such that $\lambda^i(x) > \chi > 0 > -\chi > \lambda^{i+1}(x)$ μ-a.e.*
2. *$\lim_k J^u(\mu_k)$ exists and is equal to $J^u(\mu)$*

Applying this to $f \in \mathrm{Diff}^\infty(M^2)$:

Newhouse USC. Since $r = \infty$, $h(\mu) = h_{\mathrm{top}}(f)$

Ruelle inequality. Since $d = 2$, μ-ae $\lambda^1(x) \geq h_{\mathrm{top}}(f) \geq -h_{\mathrm{top}}(f) \geq \lambda^2(x)$

B-Crovisier-Sarig (Invent. Math. 2022). If $r = \infty$ and $d = 2$, $\lim_k \lambda^1(\mu_k) = \lambda^1(\mu)$

Corollary (B-Crovisier-Sarig). Any $f \in \mathrm{Diff}^\infty(M^2)$ with $h_{\mathrm{top}}(f) > 0$ is SPR

MMEs of SPR diffeomorphisms: Lyapunov exponents

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^2)$ *of a closed surface with* $h_{\text{top}}(f) > 0$

f *is SPR if and only if the following holds:*

(*) For any $\mu_k \in \mathbb{P}_{\text{erg}}(f)$, $h(\mu_k) \to h_{\text{top}}(f)$ *and* $\exists \lim_n \mu_n =: \mu$

$\lim_k \lambda^1(\mu_k)$ *exists and is equal to* $\lambda^1(\mu)$

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{\infty}(M^2)$ *of a closed surface*

Assume $h_{\text{top}}(f) > 0$ *and* $\exists$ *a unique MME* m

If f *is SPR, there is* $C > 0$ *such that for all* $\mu \in \mathbb{P}_{\text{erg}}(f)$,

(**) $$|\lambda^1(\mu) - \lambda^1(m)| \leq C\sqrt{h_{\text{top}}(f) - h(\mu)}$$

Remark

The above is a rigidity result: if, when the entropy converges to $h_{\text{top}}(f)$*, the exponent is continuous (*) then it must be* $\frac{1}{2}$*-Hölder continuous (**).*

Sketch of proof: from continuity of exponents to SPR

$\widehat{M} := \{(x, E) : x \in M,\ E$ 1-dim subspace of $T_xM\}$ and $\widehat{f}(x, E) = (f(x), D_xf(E))$
$\widehat{\varphi} : \widehat{M} \to \mathbb{R}$ $\widehat{\varphi}(x, E) := \log |D_xf|E|$
$\mu \in \mathbb{P}(f)$ hyperbolic of saddle type (a.e. $\lambda^1(x) > 0 > \lambda^2(x)$)

1. The ergodic lifts of hyperbolic $\mu \in \mathbb{P}_{\text{erg}}(f)$ to $\mathbb{P}_{\text{erg}}(\widehat{f})$ are
$\widehat{\mu}^+ := \int_M \delta_{(x,E_x^+)}\, d\mu(x)$ and $\widehat{\mu}^- := \int_M \delta_{(x,E_x^-)}\, d\mu(x)$

2. The ergodic lifts $\widehat{\mu}$ of μ to $\mathbb{P}(\widehat{f})$ are
$\widehat{\mu} = \int (1 - a(\xi))\widehat{\mu}_\xi^+ + a(\xi)\widehat{\mu}_\xi^-\, d\xi$ if $\mu = \int \mu_\xi\, d\xi$ ergodic decompositon
Moreover, $\lambda^1(\mu) = \widehat{\mu}^+(\widehat{\varphi}) > 0$ and $\lambda^2(\mu) = \widehat{\mu}^-(\widehat{\varphi}) < 0$

3. If $\lim_k \lambda^1(\mu_k) = \lambda^1(\mu) > 0$ then $\widehat{\mu}_k^+ \xrightarrow{*} \widehat{\mu}^+$ (Computation)

4. $\forall \alpha > 0\ \exists h < h_{\text{top}}(f)\ \exists N \geq 1\ \forall \mu \in \mathbb{P}_{\text{erg}}(f)$
$h(\mu) > h \implies \mu\left(\{x \in M : \|Df^N|E_x^+\| > e^{N\cdot h_{\text{top}}(f)/2}\}\right) > 1 - \alpha$

5. $\forall \epsilon > 0\ \exists h < h_{\text{top}}(f)\ \exists C > 0\ \forall \mu \in \mathbb{P}_{\text{erg}}(f)$
$h(\mu) > h \implies \mu\left(\Lambda_{h_{\text{top}}(f)/3}(C, \epsilon)\right) > 1 - \epsilon$
Pliss Lemma. ϵ-tempered envelope.

Sketch of proof: Statistical properties, effective intrinsic ergodicity

Markov shift defined by a countable directed graph $G=(V,E)$:

$$\Sigma := \{\alpha \in V^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ (\alpha_n,\alpha_{n+1}) \in E\} \text{ with } \sigma : (\alpha_n)_{n\in\mathbb{Z}} \mapsto (\alpha_{n+1})_{n\in\mathbb{Z}}$$

Definition (Vere-Jones,Gurevič,Sarig,...)

The Markov shift Σ is **SPR** if there is $V_0 \Subset V$, $h_0 < h_{TOP}(\sigma)$, $\tau_0 > 0$ such that

$$\forall \mu \in \mathbb{P}_{\text{erg}}(\sigma)\ h(\mu) > h_0 \implies \mu(\{\alpha \in \Sigma : \alpha_0 \in V_0\}) > \tau_0$$

Theorem (Bowen, Cyr-Sarig, Gouëzel, Parry-Pollicott, Ruelle, Ruhr-Sarig, Sinai,...)

MMEs of an SPR Markov shift have "good statistical properties"

Theorem (Sarig, Benovadia)

For all $f \in \text{Diff}^{1+}(M^d)$ and $\chi > 0$:

$\exists$ Markov shift (Σ,σ) and $\pi : \Sigma \xrightarrow{C^\alpha} M$ $f\circ\pi = \pi\circ\sigma$ with "good properties" for $\mathbb{P}_\chi(f)$

Lemma. $C(x) :=$ Oseledets-Pesin; $q(x) := \limsup_{n\to\infty} e^{-2\epsilon|n|}\|C(f^n(x))^{-1}\|$

$$\forall K > 0\ \{\alpha_0 : \alpha \in \Sigma^\#,\ \pi(\alpha) = x \text{ with } q(x) < K\} \text{ is finite}$$

Lemma. If $x \in \Lambda_\chi(C,\epsilon)$, then $q(x) \le q_0(\chi,\epsilon,C)$

Theorem. If f is SPR then it has good statistical properties wrt its MMEs

Conclusion: Strong Positive Recurrence

Summary

- A notion for $\mathrm{Diff}^{1+}(M^d)$ giving:
  - existence and finiteness of MMEs
  - effective intrinsic ergodicity
  - good statistical properties of MMEs
- characterized in terms of exponents of measures with $h(\mu) \to h_{\mathrm{top}}(f)$
- It holds for all $\mathrm{Diff}^\infty(M^2)$ with $h_{\mathrm{top}}(f) > 0$

Comments

- generalization to equilibrium measures for $\psi \in C^+(M)$: yes, SRB???
- C^r smoothness for measures with $h(\mu) > \log \mathrm{Lip}(f)/r$??? (see Burguet)
- Entropy-hyperbolic diffeomorphisms in high dimension???
- Alternate route via anisotropic Hilbert spaces???

Thank you

Dziękuję .