Entropy and classification in dynamics

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Outline of the talk

1. A genealogy
2. Topological and Measure dynamics
3. Dynamical entropies
4. MMEs for smooth surface diffeos
5. Borel classification of surface diffeos
6. The hidden player: hyperbolicity
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A genealogy: Entropy from physics to dynamics

In physics:
- **Rudolf Clausius** (1865): \( S = S_0 + \int \frac{dQ}{T} \) for a fluid
- **Ludwig Boltzmann** (1872): \( H \)-theorem (1872) and \( S = k \log \Omega \) (1877)
- **Josiah W. Gibbs** (1902): \( S = -k \sum_i p_i \log p_i \) for a statistical ensemble

In probability theory, partial differential equations,…

In information theory:
- **Claude Shannon** (1948): \( H = -\sum_i p_i \log p_i \) for iid process \( X \)

In dynamics:
- **Kolmogorov-Sinaï** (1958): \( h_\mu(f) \) in ergodic theory
- **Adler-Konheim-McAndrew** (1965): \( h_{\text{top}}(f) \) in topological dynamics
- **Donald Ornstein** (1971): classification of Bernoulli schemes by entropy
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Topological dynamics

**Definition**

\((X, f)\) topological dynamical system: self-homeo \(f\) on compact metric space \(X\)

**Irreducible**

**Topologically transitive:** \(\exists x \in X \ \{f^n(x) : n \geq 0\} = X\)

**Isomorphism**

\((X, f)\) and \((Y, g)\) topologically conjugate: \(\exists h : X \to Y\) homeo st \(h \circ f = g \circ h\)

Classical examples:

**Full shift:** \(\sigma : \{1, \ldots, N\}^\mathbb{Z} \to \{1, \ldots, N\}^\mathbb{Z}\) defined by \(\sigma : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}\)

**Toral auto:** \(T_A : \mathbb{T}^2 \to \mathbb{T}^2\) defined by \(T_A(x + \mathbb{Z}^2) = Ax + \mathbb{Z}^2\) where \(A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\)

**Hénon:** \(H_{a,b} : \mathbb{S}^2 \to \mathbb{S}^2\) compactification of \((x, y) \mapsto (y + 1 - ax^2, bx)\)
Ergodic theory

**Definition**

\[(\mu, f) \text{ probabilistic dynamical system :}
\]

- a standard probability space \((X, \mathcal{X}, \mu)\)
- an automorphism: \(f : X \to X\) bimeasurable with \(f_* (\mu) := \mu \circ f^{-1} = \mu\)

**Isomorphism**

\((\mu, f) \text{ and } (\nu, g) \text{ measure conjugate:}
\]

\[\exists h : X \to Y \text{ bimeasurable st } h \circ f = g \circ h \text{ and } h_* (\mu) = \nu\]

**Irreducible**

**Ergodic:** no splitting \(X = Y \cup (X \setminus Y)\) with \(f^{-1}(Y) = Y\) and \(0 < \mu(Y) < 1\)

Toral automorphism: \((\mathbb{T}^2, \text{Leb}, T_A)\) (with Borel \(\sigma\)-algebra)

Bernoulli: \((\mathbb{N}^\mathbb{Z}, P^\mathbb{Z}, \sigma)\) where \(P\) proba on \(\mathbb{N}\), \(\sigma : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}\)

**Theorem (Birkhoff 1932)**

\[(\mu, f) \text{ ergodic probabilistic dynamical system and } \phi \in L^1(\mu)\]

For \(\mu\)-a.e. \(x\), \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k (x) = \int_X \phi \, d\mu\)
From topological to ergodic

\((X, f)\) topological dynamical system

\[ P(X) := \{\mu : X \to [0, 1] : \text{Borel probability measure}\} \]

\[ P(f) := \{\mu \in P(X) : f_*(\mu) = \mu\} \text{ and } P_{\text{erg}}(f) := \{\mu \in P(f) : \mu \text{ is } f\text{-ergodic}\} \]

**Theorem (Borel ergodic decomposition)**

\[ \exists (X_\mu)_{\mu \in P_{\text{erg}}(f) \cup \{0\}} \text{ a partition of } X \text{ into Borel sets and for any } \mu \in P_{\text{erg}}(f): \]

1. \( \mu(X_\mu) = 1 \)

2. the weak \( \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} \) exists and is \( \mu \)

For any \( \mu \in P(f) \), there is a unique \( P \in P(P_{\text{erg}}(f)) \) st \( \mu = \int_{P_{\text{erg}}(f)} \nu \, dP \)

\( X \) splits into “a parallel universes” \( X_\mu \) each described by a different \( \mu \in P_{\text{erg}}(f) \)

**Remarks.**

1) \( P_{\text{erg}}(f) \) often uncountable; some \( X_\mu \) can be finite, have positive volume, …

2) the set \( X_0 \) “beyond statistics” is null wrt any \( \mu \in P(f) \)

but can be fat in the sense of Baire, have large dimension, positive volume,…
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Dynamical entropies as counting orbit segments

As reformulated by Bowen and Dinaburg:

**Definition (Adler-Konheim-McAndrew 1968)**

The topological entropy is

\[ h_{\text{top}}(f) := \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s_f(\epsilon, n) \]

where

\[ s_f(\epsilon, n) := \max \{ \# S \subset X : x \neq y \in S \implies \exists 0 \leq k < n \, d(f^k x, f^k y) \geq \epsilon \} \]

**Remark.** \( h_{\text{top}}(f) \leq \dim_B X \cdot \log^+ \text{Lip}(f) < \infty \)

**Example.** \( h_{\text{top}}(T_A) = \log \frac{3 + \sqrt{5}}{2} \) \((A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix})\)

As reformulated by Katok and Newhouse in the ergodic case:

**Definition (Kolmogorov-Sinaï 1958)**

The entropy of \( \mu \in \mathbb{P}_{\text{erg}}(f) \) is:

\[ h_\mu(f) := \lim_{\epsilon \to 0^+} \inf_{Y: \mu(Y) > 0} \limsup_{n \to \infty} \frac{1}{n} \log s_f(\epsilon, n, Y) \]

with

\[ s_f(\epsilon, n, Y) := \max \{ \# S \subset Y : x \neq y \in S \implies \exists 0 \leq k < n \, d(f^k x, f^k y) \geq \epsilon \} \]

**Remark.** \( h_\mu(f) \leq h_{\text{top}}(f) \)

**Example.** \( h_{\mathbb{P}^\mathbb{Z}}(\sigma) = \sum_n -P(n) \log P(n) \) (Bernoulli scheme of \( P \in \mathbb{P}(\mathbb{N}) \))
(X, f) a topological dynamical system with \( \mu \in \mathbb{P}_{\text{erg}}(f) \)

\( \mathcal{P} \) partition of \( X \) into measurable sets and \( \mathcal{P}(x) := \) element of \( \mathcal{P} \) containing \( x \)

Fix \( \mu \in \mathbb{P}_{\text{erg}}(\sigma) \)

1. \( h_\mu(f) \) is the supremum of the Shannon entropies of the processes

\[
X_\mathcal{P}^n : (X, \mu) \to \mathcal{P} \text{ defined by } X_\mathcal{P}^n(x) := \mathcal{P}(\sigma^n x)
\]

ie average information revealed by \( \mathcal{P}(x) \) knowing \( (\mathcal{P}(f^{-n} x))_{n>0} \)

2. For Bernoulli schemes, \( h_\mu(f) \) is the thermodynamical entropy per site of the translation-invariant macrostate \( \mu \) of a one-dimensional spin system
Some characterizations of Kolmogorov-Sinai entropy

$(X, f)$ a topological dynamical system with $\mu \in \mathcal{P}_{\text{erg}}(f)$

$\mathcal{P}$ partition of $X$ into measurable subsets
- $\mathcal{P}^n := \{A_0 \cap f^{-1}A_1 \cap \ldots f^{-n+1}A_{n-1} : A_0, \ldots, A_{n-1} \in \mathcal{P}\}$
- $\mathcal{P}^n(x) :=$ element of $\mathcal{P}^n$ containing $x \in X$

$\mathcal{P}$ is a $\mu$-generator if $\exists X' \subset X$ s.t. $\mu(X') = 1$ and

$$x \neq y \in X' \implies \exists n \in \mathbb{Z} \mathcal{P}(f^n x) \neq \mathcal{P}(f^n y)$$

1. The **Shannon-MacMillan-Breiman theorem** gives a local formula:

$$h_{\mu}(f) = \sup_{\mathcal{P} \text{ finite}} \lim_{n \to \infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(x)) \text{ for } \mu\text{-ae } x \in X$$

2. By the **Jewett-Krieger generator theorem**, TFAE for every $N \in \mathbb{N}^*$
   - a) $\exists$ a $\mu$-generator $\mathcal{P}$ with $\#\mathcal{P} \leq N$
   - b) $h_{\mu}(f) < \log N$ (or $h_{\mu}(f) = \log N$ with $(\mu, f)$ measure conjugate to the uniform Bernoulli)

**Remark.** $h_{\mu}(f^k) = |k| \cdot h_{\mu}(f)$ for all $k \in \mathbb{Z}$
Variational principle for entropy

**Theorem (Goodman and Dinaburg)**

For any topological dynamical system, $h_{\text{top}}(f) = \sup \{ h_\mu(f) : \mu \in \mathbb{P}_{\text{erg}}(f) \}$

**Remark.** wrt entropy, $\mu \in \mathbb{P}_{\text{erg}}(f)$ approximates $X$ despite $X_0$

focus on measures with positive entropy, with large entropy, or:

**Definition**

An MME is $\mu \in \mathbb{P}_{\text{erg}}(f)$ such that $h_\mu(f) = \sup_{\nu \in \mathbb{P}_{\text{erg}}(f)} h_\nu(f)$

**Remarks.**

1) $\mu \in \mathbb{P}_{\text{erg}}(f)$ is an MME iff $h_\mu(f) = h_{\text{top}}(f)$ (variational principle)

2) $\forall r < \infty \exists$ a $C^r$-diffeomorphism without MME (Misiurewicz 1973, Buzzi 2014).

**Theorem (Newhouse 1989)**

If $f$ $C^\infty$ diffeo of a compact manifold, $\text{MME}(f) := \{ \mu \in \mathbb{P}_{\text{erg}}(f) : \text{MME} \} \neq \emptyset$

**Proof.** By analysis: $\mu \in \mathbb{P}(f) \mapsto h_\mu(f)$ is usc from Yomdin’s theory
MMEs for smooth diffeomorphisms of compact surfaces

\[ f \text{ diffeo of a compact surface } M \text{ and assume } h_{\text{top}}(f) > 0 \]

**Conjecture (Newhouse 1990)**

If \( f \) is \( C^\infty \), then \( \text{MME}(f) \) finite

**Inspiration.** Hofbauer’s 1980 result for interval maps with finite critical set

**Remark.** False in higher dimensions or finite smoothness

**Theorem (Buzzi 1997).** Newhouse’s conjecture is true for interval maps

**Theorem (Sarig 2013)**

If \( f \) is \( C^2 \), then \( \text{MME}(f) \) at most countable

\[ \forall \mu \in \text{MME}(f), (\mu, f) \text{ measure conjugate to a Bernoulli scheme } \times \text{ finite rotation} \]

**Proof.** Pesin nonuniformly hyperbolic theory and symbolic dynamics by Markov shifts ("non compact SFT")

**Theorem (Burguet 2020)**

If \( f \) is \( C^\infty \), periodic points with some hyperbolicity equidistribute to MMEs
We recently proved Newhouse’s conjecture:

**Theorem (Buzzi-Crovisier-Sarig 2022)**

$f$ a $C^\infty$ diffeo of a compact surface with $h_{\text{top}}(f) > 0$. Then $\text{MME}(f)$ is finite.

Moreover,

1) If $f$ is top. transitive (dense orbit), $\# \text{MME}(f) = 1$

2) If $f$ is top. mixing, the unique MME is conjugate to Bernoulli

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$f : M \to M$ is topologically mixing if:

\[ \forall U, V \text{ nonempty and open } \exists N \forall n \geq N U \cap f^{-n}(V) \neq \emptyset \]

**Proof.**

— Use Sarig’s symbolic dynamics
— Irreducibility of symbolic dynamics of homoclinic classes of hyperbolic periodic
— Use local uniqueness for irreducible symbolic dynamics
— Use Yomdin’s theory, Sard’s lemma and plane topology
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Ornstein’s classification of Bernoulli schemes

**Definition**

**ISOMORPHISM**

$(\mu, f)$ and $(\nu, g)$ **measure conjugate** if:

\[ \exists h : X \to Y \text{ bimeasurable st } h \circ f = g \circ h \text{ and } h_* (\mu) = \nu \]

**Lemma**

If $(\mu, f)$ and $(\nu, g)$ are measure conjugate then $h_\mu (f) = h_\nu (g)$

Bernoulli: $(\mathbb{N}^\mathbb{Z}, P^\mathbb{Z}, \sigma)$ st $P$ proba on $\mathbb{N}$, $\sigma : (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$

**Theorem (Ornstein 1970)**

*Bernoulli schemes $(P^\mathbb{Z}, \sigma)$ and $(Q^\mathbb{Z}, \sigma)$ are measure conjugate iff $h_{P^\mathbb{Z}}(\sigma) = h_{Q^\mathbb{Z}}(\sigma)$*
Definition

\( f : M \to M \) and \( g : N \to N \) are \textit{Borel conjugate} if \( \exists h : M \to N \) invertible and bi-measurable st \( h \circ f = g \circ h \)

Using Hochman’s strengthening of the Jewett-Krieger theorem

Theorem (Boyle-Buzzi 2017, Buzzi-Crovisier-Sarig 2022)

Consider \( C^\infty \) diffeos of compact surfaces with \( h_{\text{top}} > 0 \) and \( \text{top.mixing} \)

For any two such diffeos \( f : M \to M \), \( g : N \to N \) TFAE:

\( a) \ h_{\text{top}}(f) = h_{\text{top}}(g) \)

\( b) \ f|_{M \setminus \text{per}(f)} \text{ and } g|_{N \setminus \text{per}(g)} \text{ are Borel conjugate} \)
Corollary

Consider $C^\infty$ diffeos $f$ of compact surfaces such that:

1) $h_{\text{top}}(f) > 0$  
2) topologically mixing

Any two such diffeos $f : M \to M, g : N \to N$ are Borel conjugate if and only if:

i) $h_{\text{top}}(f) = h_{\text{top}}(g)$

ii) for each $p \geq 1$, $#\{x \in M : f^p(x) = x\} = #\{y \in N : g^p(y) = y\}$

Only atomic measures and MMEs are distinctive —everything in between is junk!

Remark. They are anticlassification results of Foreman, Rudolph, Weiss,…
Borel classification of smooth diffeos of compact surfaces

Without the top.mixing we need more invariant

**Definition**

Given $\mu \in \mathbb{P}_{\text{erg}}(f)$, $\text{per}(\mu) := \sup\{n \geq 1 : \exists A \text{ st } \mu(A) = 1/p \text{ and } f^p(A) = A\}$

For each $p \geq 1$,

$$h(f, p) = \sup\{h_\mu(f) : \text{per}(\mu) = p\} \text{ and }$$

$$\text{MME}(f, p) := \{\mu \in \mathbb{P}_{\text{erg}}(f) : h_\mu(f) = h(f, p)\}$$

**Example.** $\text{per}(\text{Bernoulli} \times (\mathbb{Z}/p\mathbb{Z})) = p$.

**Theorem**

Consider $C^\infty$ diffeos of compact surfaces and with positive topological entropy

For any two such diffeos $f : M \to M$, $g : N \to N$, TFAE

a) for each $p \geq 1$, $h(f, p) = h(g, p)$ and $\# \text{MME}(f, p) = \# \text{MME}(g, p)$

b) $f \mid M \setminus \text{per}(f)$ and $g \mid N \setminus \text{per}(g)$ are Borel conjugate

**Remark.** The above invariants can be read off from the topological homoclinic classes of defined by the hyperbolic periodic orbits
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$f \in \text{Diff}^1(M)$ with arbitrary Riemannian structure on $M$

**Definition (Anosov, Smale, . . .)**

An invariant (uniformly) hyperbolic set is $\Lambda \subset M$ st $T_\Lambda = E^s \oplus E^u$ and $\exists C > 0, \kappa < 1$ with

$$\forall x \in \Lambda \ \forall n \geq 0 \ \|Df^n|_{E^s_x}\| \leq C\kappa^n \text{ and } \|Df^{-n}|_{E^u_x}\| \leq C\kappa^n$$

Locally maximal if $\exists U_{\text{open}} \supset \Lambda$ st $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$

**Theorem (Sinaĭ, Parry, . . . )**

*For any locally maximal hyperbolic compact invariant $\Lambda$, $\# \text{MME}(f|\Lambda) < \infty$. Etc.*

**Proof.** Symbolic dynamics (Markov partitions, shifts of finite type)
Hyperbolicity in dynamics

\( f \) a \( C^2 \)-diffeomorphism of a \( d \)-dimensional compact manifold \( M \)

### Definition (Oseledets, Pesin)

The Lyapunov exponents of \( \mu \in \mathbb{P}_{\text{erg}}(f) \) is

\[
\lambda^1(\mu) \geq \lambda^2(\mu) \geq \cdots \geq \lambda^d(\mu)
\]

\[
\{\lambda^1(\mu), \ldots, \lambda^d(\mu)\} := \{\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_x f^n . v\| : v \in T_x M \setminus 0\} \text{ for } \mu\text{-a.e. } x
\]

A \( \mu \in \mathbb{P}_{\text{erg}}(f) \) is (Pesin) hyperbolic if \( 0 \notin \{\lambda^1(\mu), \ldots, \lambda^d(\mu)\} \).

### Remark

If \( d = 2 \), \( h_\mu(f) > 0 \) implies \( \mu \) hyperbolic by Ruelle’s inequality.

### Theorem (BCS 2022)

\( f \) a \( C^2 \) diffeo of a compact \( d \)-manifold

\( \mathcal{O} \) a hyperbolic periodic orbit

Then there is at most one \( \mu \in \mathbb{P}_{\text{erg}}(f) \) such that

1) \( \mu \) is hyperbolic and \( \mu \sim \mathcal{O} \): “homoclinically related” to \( \mathcal{O} \)
2) \( h_\mu(f) = \sup \{h_\nu(f) : \nu \sim \mathcal{O}\} \)

Moreover, \( \mu \) is measure conjugate to Bernoulli \( \times \) finite rotation.
Conclusion

Borel dynamics allows to separate Hyperbolic vs Nonhyperbolic dynamics

Ideology:
- Hyperbolic dynamics \( \approx \) rigid MMEs + standard junk + periodic orbits
- Nonhyperbolic dynamics completely different

OK for surfaces where \( h > 0 \) \( \equiv \) hyperbolicity

NOT SO MUCH in higher dimensions (except for some special classes)

THANK YOU!