Strong Positive Recurrence for Diffeomorphisms

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November 3rd, 2023

About Todd Fisher: from mathematics

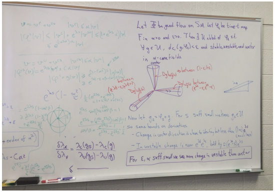
Theme: nature of entropy of diffeomorphisms and role of dominated splittings

- when is the entropy locally constant? When robustly unstable?
- what are its local/global and flexible/rigid sources?

- 2007: both working with Mike Boyle at Maryland (entropy vs partial hyperbolicity)
- 2008: first visit to Utah
- 2012: with M. Sambarino, C. Vasquez, *MMEs for certain partially hyperbolic, derived from Anosov systems,* ETDS, 17pp
- 2013: *Entropic stability beyond partial hyperbolicity,* JMD, 26pp
- 2017: with S. Crovisier, *Local perturbations of conservative C^1 diffeomorphisms,* Nonlinearity, 24pp
- 2018: with S. Crovisier, *The entropy of C^1 diffeomorphisms without a dominated splitting,* 50pp
- 2021: with K. Burns, N. Sawyer, *Phase transitions for the geodesic flow of a rank one surface with nonpositive curvature,* 9pp
- 2022: with A. Tahzibi, *A dichotomy for MMEs near time-one maps of transitive Anosov flows,* 34pp
- 2022: last visit to Paris

About Todd Fisher: to friendship

Working with Todd:



with his family:

Outline

1. Exponential mixing for MMEs of surface diffeomorphisms
2. SPR property for diffeomorphisms
3. Statistical properties of SPR diffeomorphisms
4. MMEs of SPR diffeomorphisms
5. Sketch of proofs
6. Conclusion

Spectral gap: Exponential mixing for smooth surface diffeomorphisms

$f \in \mathrm{Diff}^r(M)$ with M a closed d-dimensional manifold and $r > 1$

Classical variational principle: $h_{\mathrm{top}}(f) = \sup_{\mu \in \mathbb{P}_{\mathrm{erg}}(f)} h(\mu)$

A measure maximizing the entropy (MME) is $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(\mu) = h_{\mathrm{top}}(f)$

If $r = \infty$: $\exists$ MME exists (Newhouse)
If $r = \infty$, $d = 2$, $h_{\mathrm{top}}(f) > 0$, top. mixing: $\exists!$ Bernoulli MME (B-Crovisier-Sarig)

Theorem (B.-Crovisier-Sarig)

$f \in \mathrm{Diff}^\infty(M^{d=2})$ *topological mixing and* $h_{\mathrm{top}}(f) > 0$.

Then the MME μ *is exponentially mixing for Hölder functions, ie*

For any $0 < \alpha \leq 1$, *there is* $\kappa < 1$ *such that*

$$\forall u, v \in C^\alpha(M) \int u \circ f^n \cdot v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)$$

Proof

SPR as a new notion of hyperbolicity

- sufficiently relaxed to hold beyond uniform hyperbolicity
- strong enough to retain "spectral gap" properties

Nonuniformly hyperbolic measures

$f \in \mathrm{Diff}^r(M^d)$ with $r > 1$ and M a closed Riemannian manifold and $d \geq 2$

$\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ ergodic (invariant Borel probability) measure

Definition (Oseledets)

Lyapunov spectrum: $\sigma_L(x) := \{\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n.v\| : v \in T_x M \setminus 0\}$

Oseledets spaces: $E_x^\lambda := \{v \in T_x M \setminus 0 : \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| = \lambda\} \cup \{0\}$

Lyapunov exponents: $\lambda^1(x) \geq \lambda^2(x) \geq \cdots \geq \lambda^d(x)$ $\lambda^j(\nu) := \int \lambda^j(x)\, d\nu$ for $\nu \in \mathbb{P}(f)$

Definition (Pesin hyperbolicity)

$\nu \in \mathbb{P}(f)$ is **hyperbolic** if ν-ae $0 \notin \sigma_L(x)$

$\nu \in \mathbb{P}(f)$ is hyperbolic of **saddle type** if additionally ν-ae $\lambda^1(x) > 0 > \lambda^d(x)$

Key observation (Katok)

If $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$, $h(\mu) > 0$ and $d = 2$, then (Ruelle) μ hyperbolic saddle type

More precisely, $\lambda^1(\mu) \geq h(\mu) > 0 > -h(\mu) \geq \lambda^2(\mu)$

Pesin theory of nonuniform hyperbolicity

Definition

For $0 < \epsilon < \chi$ and $C \geq 1$, the **Pesin block** $\Lambda_\chi(C, \epsilon)$ is the set of $x \in M$ for which there is $T_x M = E \oplus F$ satisfying:

$$\forall n \geq 0 \, \forall k \in \mathbb{Z} \quad \|Df^n|_{Df^k(E)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$
$$\forall n \geq 0 \, \forall k \in \mathbb{Z} \quad \|Df^{-n}|_{Df^k(F)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$

Fact

$\Lambda_\chi(C, \epsilon)$ is compact with $f^{\pm 1}(\Lambda_\chi(C, \epsilon)) \subset \Lambda_\chi(Ce^\epsilon, \epsilon)$

Invariant $K \Subset M$ is hyperbolic iff $K \subset \Lambda_\chi(C, \epsilon)$ for some $\chi > 0$ and arb.small $\epsilon > 0$

Lemma (Oseledets-Pesin reduction)

If $\mu \in \mathbb{P}(f)$ χ-hyperbolic ie μ-ae $\{\lambda^j(x) : j = 1, \ldots, d\} \cap [-\chi, \chi] = \emptyset$ then

$$\forall \epsilon > 0 \; \exists C > 1 \; \mu(\Lambda_\chi(C, \epsilon)) > 1 - \epsilon$$

Theorem (Pesin local invariant manifolds)

For $\chi > 0$ and $0 < \epsilon \leq \epsilon(f, \chi)$, $W^s(x), W^u(x)$ are C^r-immersions, C^0 on $x \in \Lambda_\chi(C, \epsilon)$

SPR property for diffeomorphisms

$f \in \mathrm{Diff}^r(M^d)$ with $r > 1$ and M closed d-dimensional manifold

Definition

For $0 < \epsilon < \chi$ and $C \geq 1$, the Pesin block $\Lambda_\chi(C, \epsilon)$ is the set of $x \in M$ for which there is $T_x M = E \oplus F$ satisfying:

$$\forall n \geq 0 \; \forall k \in \mathbb{Z} \quad \|Df^n|_{Df^k(E)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$
$$\forall n \geq 0 \; \forall k \in \mathbb{Z} \quad \|Df^{-n}|_{Df^k(F)}\| \leq Ce^{\epsilon|k|} \exp(-\chi n)$$

Definition (B-Crovisier-Sarig)

$f \in \mathrm{Diff}^r(M^d)$ is **strongly positively recurrent** (or **SPR**) if there is $\chi > 0$ such that

For any $\epsilon > 0$, there are $h < h_{\mathrm{top}}(f)$ and $C, \tau > 0$ such that

$$\forall \mu \in \mathbb{P}_{\mathrm{erg}}(f) \;\; h(\mu) > h \implies \mu(\Lambda_\chi(C, \epsilon)) > \tau$$

Statistical properties

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^d)$ be SPR with MME μ ($\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ and $h(\mu) = h_{\mathrm{top}}(f)$)

If μ is strongly mixing then it is exponentially mixing *for Hölder functions, ie*

For any exponent $0 < \alpha \leq 1$, there is $\kappa < 1$ such that

$$\forall u, v \in C^\alpha(M) \int u \circ f^n \cdot v \, d\mu - \int_M u \, d\mu \int_M v \, d\mu = O(\kappa^n)$$

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^d)$ be SPR with $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$ such that $h(\mu) = h_{\mathrm{top}}(f)$

If μ is strongly mixing then, for Hölder-continuous functions we have:

- *Central limit theorem (CLT)*
- *Identification of the variance and characterization of its vanishing*
- *Large deviations*
- *Almost sure invariance principle and its consequences (law of iterated logarithm, arcsine law, law of records)*

Remark. There are statements for general (ie periodic) ergodic MMEs

MMEs of SPR diffeomorphisms: existence and finiteness

Theorem (B-Crovisier-Sarig)

Any SPR $f \in \mathrm{Diff}^{1+}(M^d)$ *admits a nonzero,finite number of ergodic MMEs*

Proof.

Finiteness:

- From BCS' irreducible symbolic dynamics, each MME belongs to a distinct equivalence class for *homoclinic relation*
- SPR implies all MMEs see a common Pesin set
- A Pesin set can meet only finitely many measured homoclinic classes

Existence:

- Show that Sarig/Ben Ovadia's coding is SPR as a Markov shift
- It follows it has some MME
- Project it to the diffeo

□

Remark

For $d = 2$ *new proof of the finiteness and existence of MMEs*

SPR property and Lyapunov exponents

$\lambda^1(x) \geq \cdots \geq \lambda^d(x)$: pointwise Lyapunov exponents (repeated according to multiplicity)
$\lambda^j(\mu) := \int_X \lambda^j(x)\, d\mu$: average Lyapunov exponents (repeated according to multiplicity)
$J^u(\mu) := \sum_{j=1}^d \max(\lambda^j(\mu), 0)$: sum of positive exponents

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^d)$ of a closed manifold

f is SPR if and only if $\exists \chi > 0$ such that the following holds:

For any $\mu_k \in \mathbb{P}_{\mathrm{erg}}(f)$ with $h(\mu_k) \to h_{\mathrm{top}}(f)$ and $\exists \lim_n \mu_n =: \mu$

1. *$\exists i = i(\mu)$ such that $\lambda^i(x) > \chi > 0 > -\chi > \lambda^{i+1}(x)$ μ-a.e.*
2. *$\lim_k J^u(\mu_k)$ exists and is equal to $J^u(\mu)$*

Theorem (B-Crovisier-Sarig)

Any $f \in \mathrm{Diff}^{1+}(M^2)$ SPR with $h_{\mathrm{top}}(f) > 0$ with unique MME m satisfies:

There is $C > 0$ such that for all $\mu \in \mathbb{P}_{\mathrm{erg}}(f)$,

$$|J^u(\mu) - J^u(m)| \leq C\sqrt{h_{\mathrm{top}}(f) - h(\mu)}$$

SPR property and Lyapunov exponents

For $d = 2$ using Ruelle's inequality:

Theorem (B-Crovisier-Sarig)

Let $f \in \mathrm{Diff}^{1+}(M^2)$ of a closed surface with $h_{\mathrm{top}}(f) > 0$

f is SPR if and only if the following holds:

(*) For any $\mu_k \in \mathbb{P}_{\mathrm{erg}}(f)$, $h(\mu_k) \to h_{\mathrm{top}}(f)$ and $\exists \lim_n \mu_n =: \mu$

$\lim_k \lambda^1(\mu_k)$ *exists and is equal to* $\lambda^1(\mu)$

Theorem (B-Crovisier-Sarig (Invent. Math. 2022))

For $f \in \mathrm{Diff}^{\infty}(M^2)$, if $\lim_k h(\mu_k) = h_{\mathrm{top}}(f) > 0$ then

$h(\mu) = h_{\mathrm{top}}(f)$ *and* $\lim_k \lambda^1(\mu_k) = \lambda^1(\mu)$

Corollary (B-Crovisier-Sarig)

Any $f \in \mathrm{Diff}^{\infty}(M^2)$ with $h_{\mathrm{top}}(f) > 0$ is SPR

Sketch of proof: Statistical properties, effective intrinsic ergodicity

Markov shift defined by a countable directed graph $G = (V, E)$:

$$\Sigma := \{\alpha \in V^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ (\alpha_n, \alpha_{n+1}) \in E\} \text{ with } \sigma : (\alpha_n)_{n\in\mathbb{Z}} \mapsto (\alpha_{n+1})_{n\in\mathbb{Z}}$$

Definition (Vere-Jones,Gurevič,Sarig,...)

The Markov shift Σ is **SPR** if there is $V_0 \Subset V$, $h_0 < h_{TOP}(\sigma)$, $\tau_0 > 0$ such that

$$\forall \mu \in \mathbb{P}_{\mathrm{erg}}(\sigma)\ h(\mu) > h_0 \implies \mu(\{\alpha \in \Sigma : \alpha_0 \in V_0\}) > \tau_0$$

Theorem (Bowen, Cyr-Sarig, Gouëzel, Parry-Pollicott, Ruelle, Ruhr-Sarig, Sinai,...)

MMEs of an SPR Markov shift have "good statistical properties"

Theorem (Sarig, Benovadia)

For all $f \in \mathrm{Diff}^{1+}(M^d)$ *and* $\chi > 0$*:*

$\exists$ *Markov shift* (Σ, σ) *and* $\pi : \Sigma \stackrel{C^\alpha}{\to} M$ $f \circ \pi = \pi \circ \sigma$ *with "good properties" for* $\mathbb{P}_\chi(f)$

Lemma. $C(x) :=$ Oseledets-Pesin; $q(x) := \limsup_{n\to\infty} e^{-2\epsilon|n|}\|C(f^n(x))^{-1}\|$

$$\forall K > 0\ \{\alpha_0 : \alpha \in \Sigma^\#,\ \pi(\alpha) = x \text{ with } q(x) > K\} \text{ is finite}$$

Lemma. If $x \in \Lambda_\chi(C, \epsilon)$, then $q(x) \geq q_0(\chi, \epsilon, C)$

Theorem. If f is SPR then it has good statistical properties wrt its MMEs

Conclusion: Strong Positive Recurrence

Summary

- SPR is a notion for $\mathrm{Diff}^{1+}(M^d)$ giving:
  - existence and finiteness of MMEs
  - good statistical properties of MMEs
- characterized in terms of exponents of measures with $h(\mu) \to h_{\mathrm{top}}(f)$
- It holds for all $\mathrm{Diff}^{\infty}(M^2)$ with $h_{\mathrm{top}}(f) > 0$

Comments

- how common is SPR in $\mathrm{Diff}^{\infty}(M^d)$, $d \geq 3$?
- C^r smoothness for measures with $h(\mu) > \log \mathrm{Lip}(f)/r$? (see Burguet)
- generalization to equilibrium measures for $\psi \in C^+(M)$: yes but SRB?